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Three-spin correlation of the free-fermion model

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Abstract. We consider the free-fermion model on the square lattice and obtain the local three-spin correlation function for three spins surrounding a unit cell of the lattice. As an application of this result, we compute the spontaneous magnetisation for the Ising model on the Union Jack lattice with the most general anisotropic interactions.

1. Introduction

The two-dimensional free-fermion model on the square lattice (Fan and Wu 1970) is a lattice model solvable by the method of Pfaffians. It can also be regarded as an interaction-around-the-face model for which the local Boltzmann factors satisfy a constraint, known as the free-fermion condition. Indeed, in the latter representation, one can treat the model as a spin system and inquire about the exact expressions of its correlation functions. Recently, Baxter (1986) obtained explicit expressions for the local one- and two-spin correlations of the free-fermion model. In this paper we show that the consideration can be extended to local three-spin correlations, and we obtain explicit expressions for the correlation of three spins located around a unit cell of the square lattice. As an application of our result, we compute the spontaneous magnetisation of the Union Jack Ising lattice with complete anisotropic interactions†.

The strategy of our analysis is to convert the free-fermion model into a checkerboard model which possesses the same three-spin correlation functions in question as the free-fermion model. Then, making use of the known spontaneous magnetisations of the checkerboard model obtained recently by us (Lin and Wu 1988) and the generalisation of a correlation identity used by Choy and Baxter (1987), we derive explicit expression for the three-spin correlations. The calculation is further facilitated by the introduction of an intermediate step of a Union Jack lattice, which provides an indispensible alternate formulation.

The organisation of this paper is as follows. In § 2 we review the definition of the free-fermion model and define the problem at hand. In § 3 we convert the free-fermion model, first to a Union Jack Ising model, then to a checkerboard Ising lattice. Calculations of the three-spin correlation functions are carried out in § 4 using the resulting checkerboard formulation, and in § 5 we apply the result of § 4 to deduce the spontaneous magnetisation of the general anisotropic Union Jack Ising lattice.

† A preliminary version of this work has been reported by one of us (KYL) at the 5th Republic of China-Republic of Korea symposium on Solid State Physics at Seoul, July 12-14, 1988.

2. Definitions

Consider an eight-vertex model with vertex configurations and weights shown in figure 1. Place Ising spins in the faces of the lattice and require two neighbouring spins to be opposite if, and only if, they are separated by a bond. Then, there exists a two-to-one mapping between the Ising spin configurations and the vertex configurations of the free-fermion model, and we can equally well describe the vertex weights by specifying the spins surrounding each lattice site.

Let a, b, c, d be the four spins surrounding a lattice site in the arrangement shown in figure 1. Then the eight vertex weights $\omega(a, b, c, d) = \omega(-a, -b, -c, -d)$ are

$$\begin{aligned} &\omega_1 = \omega(++++) && \omega_2 = \omega(+-+-) \\ &\omega_3 = \omega(+--+) && \omega_4 = \omega(++--) \\ &\omega_5 = \omega(+-++) && \omega_6 = \omega(+++-) \\ &\omega_7 = \omega(++-+) && \omega_8 = \omega(-+++) \end{aligned} \tag{1}$$

and are all positive. The free-fermion model is the eight-vertex model whose vertex weights satisfy the free-fermion constraint

$$\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8. \tag{2}$$

In *any* spin configuration vertices of types (5), (6) and (7), (8) always occur in pairs. As a consequence, the weights ω_5, ω_6, ω_7, ω_8 occur only in the combination of $\omega_5\omega_6$ and $\omega_7\omega_8$, and there is no loss of generality if we replace ω_5 and ω_6 by ω, and ω_7 and ω_8 by ω', where

$$\omega = \sqrt{\omega_5\omega_6} \qquad \omega' = \sqrt{\omega_7\omega_8}. \tag{3}$$

As a result, the vertex weights (1) do not differentiate the upper right from the lower left (and upper left from lower right) directions. Particularly, there are only two independent three-spin correlation functions among the four spins a, b, c, d, namely

$$\begin{aligned} M_1 &= \langle bcd \rangle = \langle dab \rangle \\ M_2 &= \langle abc \rangle = \langle cda \rangle \end{aligned} \tag{4}$$

where

$$\langle X \rangle \equiv \frac{\Sigma\, X\, \Pi\, \omega(a, b, c, d)}{\Sigma\, \Pi\, \omega(a, b, c, d)}. \tag{5}$$

In (5), the summations are taken over all vertex, or spin, configurations and the products over all vertex sites. Our goal is to compute both M_1 and M_2 in terms of the vertex weights (1) and (3).

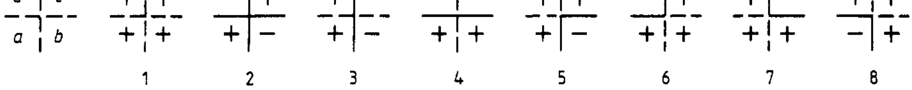

Figure 1. The eight vertex configurations and weights of the eight-vertex model.

3. Equivalence to a checkerboard lattice

It is convenient to rotate the lattice 45° in the clockwise direction so that the four spins a, b, c, d of figure 1 are arranged as shown in figure 2. We now convert the free-fermion model into a checkerboard Ising model, shown in figure 3, via the intermediate step of a Union Jack lattice of figure 2†. An important fact is that these transformations leave the three-spin correlations M_1 and M_2 unchanged and, consequently, we can choose a convenient representation to carry out the calculation.

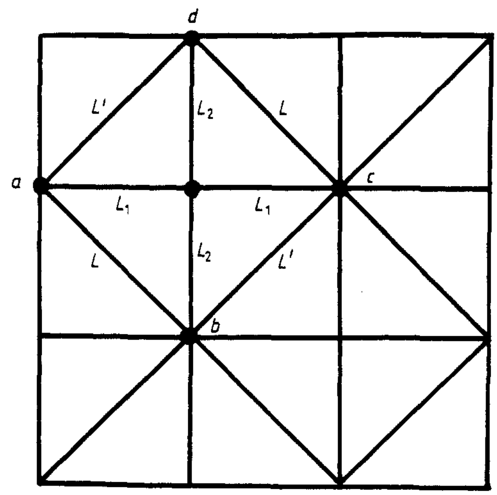

Figure 2. The Union Jack lattice.

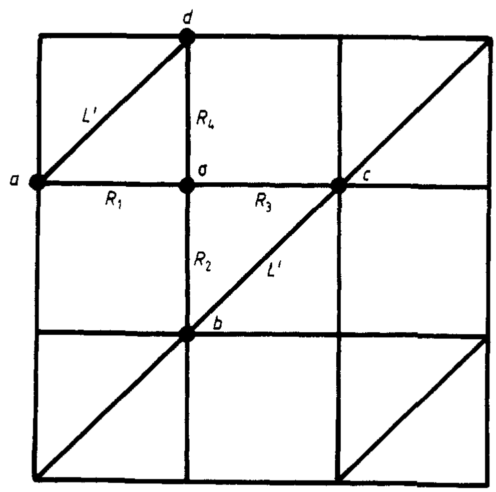

Figure 3. The checkerboard lattice.

† It is also possible to go directly from the free-fermion model to the checkerboard Ising model, but the use of the intervening Union Jack lattice makes the presentation clearer.

Our first step is to convert the free-fermion model into the Union Jack model of figure 2. This is done by writing

$$\omega(a, b, c, d) = 2\rho \exp[\tfrac{1}{2}L(ab+cd)+\tfrac{1}{2}L'(ad+bc)] \cosh[L_1(a+c)+L_2(b+d)] \tag{6}$$

which becomes, upon using (1) and (3),

$$\begin{aligned} \omega_1 &= 2\rho\, e^{L+L'} \cosh 2(L_1+L_2) \\ \omega_2 &= 2\rho\, e^{-(L+L')} \cosh 2(L_1-L_2) \\ \omega_3 &= 2\rho\, e^{-L+L'} \\ \omega_4 &= 2\rho\, e^{L-L'} \\ \omega &= 2\rho \cosh 2L_1 \\ \omega' &= 2\rho \cosh 2L_2. \end{aligned} \tag{7}$$

Only five of the six vertex weights in (7) are independent as they are related through the free-fermion condition (2). We can solve (7) for L_1, L_2, L, L' (and ρ), obtaining

$$\cosh 2L_1 = \omega/\sqrt{\omega_3\omega_4} \tag{8a}$$

$$\cosh 2L_2 = \omega'/\sqrt{\omega_3\omega_4} \tag{8b}$$

$$e^{-2L} = [\omega\omega' \pm \sqrt{(\omega_1\omega_2-\omega^2)(\omega_1\omega_2-\omega'^2)}]/\omega_1\omega_4 \tag{8c}$$

$$e^{-2L'} = [\omega\omega' \pm \sqrt{(\omega_1\omega_2-\omega^2)(\omega_1\omega_2-\omega'^2)}]/\omega_1\omega_3. \tag{8d}$$

We note that ($8a$) and ($8b$) determine L_1, L_2 up to an arbitrary sign. For the sake of easier visualisation, we explicitly consider

$$\omega_3\omega_4 < \{\omega^2, \omega'^2\} \qquad \text{or} \qquad \omega_1\omega_2 > \{\omega^2, \omega'^2\} \tag{9}$$

so that both L_1 and L_2 are real. Our final results, however, are independent of this restriction. The upper signs in ($8c$) and ($8d$) now correspond to L_1, L_2 having the same sign, e.g. $L_1 > 0$, $L_2 > 0$, and the lower signs correspond to the negation of, say, L_2, i.e. $L_1 > 0$, $L_2 < 0$. Thus, (8) offers two distinct realisations of the free-fermion model as a spin model, a fact we shall explicitly use in later considerations.

We next introduce a local star-star transformation which transforms the Union Jack lattice into the checkerboard lattice of figure 3. The star-star transformation, shown in figure 4, applies to a unit cell of the Union Jack lattice with the L' interactions

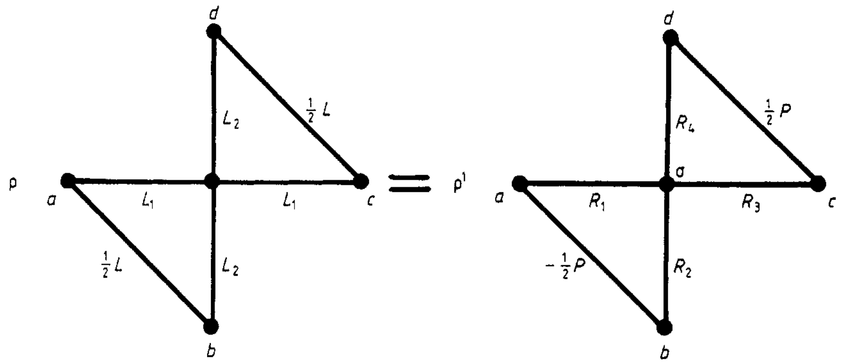

Figure 4. The star-star transformation converting the Union Jack lattice of figure 2 into the checkerboard lattice of figure 3.

deleted and thus having the Boltzmann weight

$$\begin{aligned} W(a, b, c, d) &\equiv \exp[-\tfrac{1}{2}L'(ad+bc)]\omega(a, b, c, d) \\ &= 2\rho' \exp[\tfrac{1}{2}P(cd-ab)] \cosh(R_1a+R_2b+R_3c+R_4d). \end{aligned} \tag{10}$$

Defining W_i in terms of $W(a, b, c, d)$ in the same way as ω_i is defined in (1) and (3), we can explicitly rewrite (10) as

$$\begin{aligned}
W_1 &= e^{-L'}\omega_1 = 2\rho\, e^{L} \cosh 2(L_1+L_2) &&= 2\rho' \cosh(R_1+R_2+R_3+R_4) \\
W_2 &= e^{L'}\omega_2 = 2\rho\, e^{-L} \cosh 2(L_1-L_2) &&= 2\rho' \cosh(R_1-R_2+R_3-R_4) \\
W_3 &= e^{-L'}\omega_3 = 2\rho\, e^{-L} &&= 2\rho' \cosh(R_1-R_2-R_3+R_4) \\
W_4 &= e^{L'}\omega_4 = 2\rho\, e^{L} &&= 2\rho' \cosh(R_1+R_2-R_3-R_4) \\
W &= \omega = 2\rho \cosh 2L_1 &&= 2\rho'\, e^{P} \cosh(R_1-R_2+R_3+R_4) \\
& &&= 2\rho'\, e^{-P} \cosh(R_1+R_2+R_3-R_4) \\
W' &= \omega' = 2\rho \cosh 2L_1 &&= 2\rho'\, e^{-P} \cosh(R_1+R_2-R_3+R_4) \\
& &&= 2\rho'\, e^{P} \cosh(-R_1+R_2+R_3+R_4).
\end{aligned} \tag{11}$$

Note that the W weights satisfy the free-fermion condition

$$W_1W_2+W_3W_4 = W^2+W'^2 \tag{12}$$

and the further constraint

$$W_1W_3+W_2W_4 = 2WW' \tag{13}$$

indicating that only four of the six weights are independent. As a consequence, the four interactions R_i are related, and their relationship is

$$\sinh 2R_1 \sinh 2R_4 = \sinh 2R_2 \sinh 2R_3. \tag{14}$$

The next step is to solve (11) for R_i (and P). However, (11) is identical to (2.5) of Baxter (1986) (with $M=0$ therein) for which Baxter has already obtained the solution. For completeness, we give here this solution in our notation

$$\begin{aligned}
\cosh 2P &= (W_1W_4+W_2W_3)/2WW' = (\omega_1\omega_4+\omega_2\omega_3)/2\omega\omega' \\
\cosh 2R_i &= (s^2+s_i^2-s_j^2-s_{ij}^2)/2(ss_i-s_js_{ij}) \\
\sinh 2R_i/\sinh 2R_j &= (ss_j-s_is_{ij})/(ss_i-s_js_{ij})
\end{aligned} \tag{15}$$

with

$$\begin{aligned}
s &= W_1 = e^{-L'}\omega_1 & s_{13} &= s_{24} = W_2 = e^{L'}\omega_2 \\
s_{14} &= s_{23} = W_3 = e^{-L'}\omega_3 & s_{12} &= s_{34} = W_4 = e^{L'}\omega_4 \\
s_1 &= e^{-P}W' = e^{-P}\omega' & s_2 &= e^{-P}W = e^{-P}\omega \\
s_3 &= e^{P}W' = e^{P}\omega' & s_4 &= e^{P}W = e^{P}\omega.
\end{aligned} \tag{16}$$

These relations completely determine R_i up to an overall sign in terms of the W weights and, after introducing (8d) for $e^{-2L'}$, in terms of the free-fermion weights.

4. Computation of the three-spin correlations M_1 and M_2

The transformation of the free-fermion model to the Union Jack and checkerboard lattices leaves the eight-coordinated spins a, b, c, d unchanged and, consequently, does not alter the three-spin correlations M_1 and M_2. It is then convenient to compute M_1 and M_2 using whichever representation that is convenient. We now proceed with the checkerboard lattice.

The computation is based on the use of the following identity, which is a generalisation of that used by Choy and Baxter (1987) to anisotropic interactions

$$\langle\sigma\rangle = \langle\tanh(R_1a+R_2b+R_3c+R_4d)\rangle \tag{17}$$

where $\langle\cdot\rangle$ is defined in (5), and σ is the spin connecting to a, b, c and d as shown in figures 3 and 4. Note that $\langle\sigma\rangle$ changes sign if the signs of all R_i are reversed, so we can always choose a solution of (15) to make $\langle\sigma\rangle$ positive. After expanding $\tanh(R_1a+R_2b+R_3c+R_4d)$ into a linear combination of a, b, c, d, abc, bcd, cda, dab, we find

$$\begin{aligned}\langle\sigma\rangle = {} & A^+_{1234}\langle a\rangle + A^+_{2341}\langle b\rangle + A^+_{3412}\langle c\rangle + A^+_{4123}\langle d\rangle \\ & + A^-_{1234}\langle bcd\rangle + A^-_{2341}\langle cda\rangle + A^-_{3412}\langle dab\rangle + A^-_{4123}\langle abc\rangle\end{aligned} \tag{18}$$

where

$$\begin{aligned}A^{\pm}_{ijkl} &= \tfrac{1}{8}(G_{ijkl}\pm H_{ijkl}) \\ G_{ijkl} &= \tanh(R_i+R_j+R_k+R_l)+\tanh(R_i+R_j-R_k-R_l) \\ &\quad +\tanh(R_i-R_j-R_k+R_l)+\tanh(R_i-R_j+R_k-R_l) \\ H_{ijkl} &= \tanh(R_i-R_j-R_k-R_l)+\tanh(R_i-R_j+R_k+R_l) \\ &\quad +\tanh(R_i+R_j-R_k+R_l)+\tanh(R_i+R_j+R_k-R_l).\end{aligned} \tag{19}$$

For an infinite lattice we have

$$\langle a\rangle = \langle b\rangle = \langle c\rangle = \langle d\rangle = I \tag{20}$$

where I is the spontaneous magnetisation of the free-fermion model, an explicit expression of which has been given by Baxter (1986):

$$I = (1-\Omega^{-2})^{1/8} \tag{21}$$

with

$$\Omega^2 = \frac{(s-\omega_1)(s-\omega_2)(s-\omega_3)(s-\omega_4)}{\omega^2\omega'^2}$$

where $s=\frac{1}{2}(\omega_1+\omega_2+\omega_3+\omega_4)$; I is zero whenever the quantity inside the large bracket in (21) is negative, namely, when the system is in the disordered regime. Substituting (19) and (20) into (18) and after some rearrangement of terms, we obtain

$$\langle\sigma\rangle/I = \tanh(R_1+R_2+R_3+R_4) - T_{1234}(R)z_1 - T_{2341}(R)z_2 \tag{22}$$

where

$$\begin{aligned} z_i &= M_i/I - 1 \qquad i=1,2 \\ T_{1234}(R) &= \sinh 2R_2 \sinh 2R_4 \sinh 2(R_1+R_3)/G_+(R)G_-(R) \\ G_{\pm}(R) &= \cosh 2(R_1+R_3)+\cosh 2(R_2\pm R_4).\end{aligned} \tag{23}$$

Equation (22) is the basic relation from which M_1 and M_2 are to be computed. As it stands, however, (22) contains three unknown quantities, z_1, z_2 and $\langle\sigma\rangle$. We now show that $\langle\sigma\rangle$ can be computed.

The lattice shown in figure 3 is a checkerboard lattice for which the one-spin correlation functions are known (Lin and Wu 1988). Specialising the result of Lin and Wu (1988) to figure 3, we obtain after some algebra the expression

$$\frac{\langle\sigma\rangle}{I}=\frac{\sqrt{B_+}+\sqrt{B_-}}{\sqrt{Z}} \tag{24}$$

where

$$Z=B_++B_-+2\sqrt{B_+B_-}+4(1-\mathrm{e}^{-4L'})$$
$$\begin{aligned}B_\pm=&[\cosh 2(R_1+R_4)\pm 1][\cosh 2(R_2+R_3)\mp 1]\\&-\mathrm{e}^{-4L'}[\cosh 2(R_1-R_4)\pm 1][\cosh 2(R_2-R_3)\mp 1].\end{aligned} \tag{25}$$

It is now straightforward to substitute (24) into (22), using $R_i=R_i(\{\omega\}, L')$ given by (15), where L' is given in terms of the ω by (8d). Then, (22) becomes an equation containing only ω_1, ω_2, ω_3, ω_4, ω, ω' and the two unknowns z_1 and z_2.

Now, (8d) gives L' in terms of the ω in two different ways, corresponding to $L_1>0$ and L_2 (of the Union Jack lattice) being either positive or negative, as remarked in discussions following (9). Thus, (22) generates two equations by using (8d) with the upper and lower signs, respectively. After a very lengthy yet straightforward calculation, these two equations become

$$F_\pm=A_\pm-B_1\left(\frac{M_1}{I}-1\right)\mp B_2\left(\frac{M_2}{I}-1\right) \tag{26}$$

where

$$\begin{aligned}
&F_\pm=\left(\frac{A+2\sqrt{BC}}{D+2E\sqrt{B}}\right)^{1/2} \qquad F_+F_-=\frac{\omega^2-\omega'^2}{\omega_1\omega_2}\\
&A_\pm=\tanh 2(L_1\pm|L_2|)\\
&\quad=\frac{1}{\omega_1\omega_2}(\omega\sqrt{\omega_1\omega_2-\omega'^2}\pm\omega'\sqrt{\omega_1\omega_2-\omega^2})\\
&B_1=\tfrac14[-\tanh 2(L_1+L_2)-\tanh 2(L_1-L_2)+2\tanh 2L_1]\\
&\quad=\frac{\omega_1\omega_2-\omega^2}{2\omega\omega_1\omega_2}\sqrt{\omega_1\omega_2-\omega'^2}\\
&B_2=\tfrac14[-\tanh 2(L_1+L_2)+\tanh 2(L_1-L_2)+2\tanh 2L_1]\\
&\quad=\frac{\omega_1\omega_2-\omega'^2}{2\omega'\omega_1\omega_2}\sqrt{\omega_1\omega_2-\omega^2}
\end{aligned} \tag{27}$$

with

$$\begin{aligned}
&A=2\omega^2\omega'^2(\omega_1^2+\omega_2^2+\omega_3^2+\omega_4^2)-(\omega_1\omega_2+\omega_3\omega_4)(\omega_1\omega_3+\omega_2\omega_4)(\omega_1\omega_4+\omega_2\omega_3)\\
&B=\omega^2\omega'^2(\omega^2\omega'^2-\omega_1\omega_2\omega_3\omega_4)\\
&C=(\omega_1^2+\omega_2^2-\omega_3^2-\omega_4^2)^2-4(\omega^2-\omega'^2)^2\\
&D=(\omega_1^2+\omega_2^2)(2\omega^2\omega'^2-\omega_1\omega_2\omega_3\omega_4)-\omega_1^2\omega_2^2(\omega_3^2+\omega_4^2)\\
&E=\omega_1^2-\omega_2^2.
\end{aligned} \tag{28}$$

Finally, solving (26) for M_1 and M_2, we obtain

$$\begin{aligned} M_1 &= I\left(\frac{\omega_1\omega_2+\omega^2}{\omega_1\omega_2-\omega^2}-\frac{F_+ + F_-}{2B_1}\right) \\ M_2 &= I\left(\frac{\omega_1\omega_2+\omega'^2}{\omega_1\omega_2-\omega'^2}-\frac{F_+ - F_-}{2B_2}\right). \end{aligned} \tag{29}$$

Equation (29), in which I is given by (21), gives the desired expressions for the three-spin correlations. Note that M_1 and M_2 can be obtained from each other by an interchange of ω and ω'. Note that, as remarked earlier, these results are valid for all ω, independent of the validity of (9).

In the special case of $\omega^2=\omega'^2=(\omega_1\omega_2+\omega_3\omega_4)/2$, (29) reduces to

$$M_1=M_2=I\left[\frac{3\omega_1\omega_2+\omega_3\omega_4}{\omega_1\omega_2-\omega_3\omega_4}-\frac{2\omega_1(\omega_1\omega_2+\omega_3\omega_4)}{\omega_1\omega_2-\omega_3\omega_4}\left(\frac{\omega_1^2+\omega_2^2-\omega_3^2-\omega_4^2}{(\omega_1^2-\omega_3^2)(\omega_1^2-\omega_4^2)}\right)^{1/2}\right]. \tag{30}$$

5. Spontaneous magnetisation of the anisotropic Union Jack Ising lattice

As an application of the result obtained in § 4, we now compute the spontaneous magnetisation of the Union Jack Ising lattice with general anisotropic interactions.

The most general anisotropic Union Jack lattice, shown in figure 5, possesses six distinct interactions $J_1, J_2, J_3, J_4, J, J'$. In the limit of an infinite lattice, the spontaneous magnetisation of the lattice is

$$M_0=\tfrac{1}{2}(\langle a\rangle+\langle\tau\rangle) \tag{31}$$

where $\langle a\rangle$ and $\langle\tau\rangle$ are, respectively, the per-site magnetisations of the two sublattices of eight-coordinated and four-coordinated sites. The evaluation of $\langle a\rangle$ for the general lattice is relatively simple and an expression for $\langle a\rangle$ has been given by Wu and Lin (1987) (see also discussions below). In fact, as early as two decades ago Vaks *et al* (1966) obtained the free energy and the sublattice magnetisation $\langle a\rangle$ for the symmetric

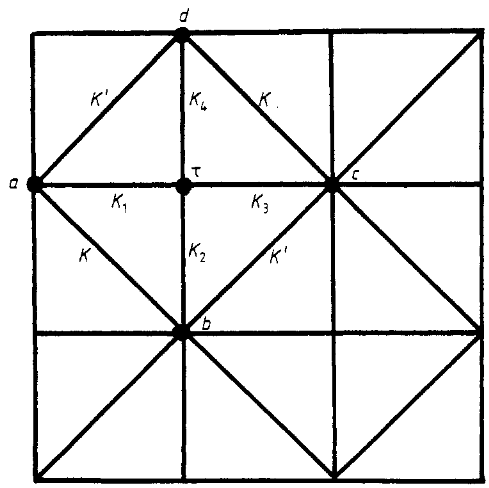

Figure 5. The Union Jack lattice with the most general anisotropic interactions.

case of $J_i = J_1$, $J = J'$; they also showed that the system exhibits a re-entry transition, a phenomenon which has since been shown to persist for $J_1 = J_2 = J_3 \neq J_4$, $J = J'$ (Sacco and Wu 1975). Interest in the Union Jack Ising system was revived recently when Lin and Wang (1987) published a conjectured form of the sublattice magnetisation $\langle \tau \rangle$ in the symmetric case. Their conjecture has since been proven to hold by Choy and Baxter (1987), and Lin and Wang (1988) further extended Choy and Baxter's derivation of $\langle \tau \rangle$ to the case of $J_i = J_1$, $J \neq J'$. We now complete the task by computing $\langle \tau \rangle$ for the most general anisotropic Union Jack lattice.

Our calculation of $\langle \tau \rangle$ is based on the use of the following identity, derived in analogy to (22) for the Union Jack lattice:

$$\frac{\langle \tau \rangle}{\langle a \rangle} = \tanh(K_1 + K_2 + K_3 + K_4) - T_{1234}(K) z_1 - T_{2341}(K) z_2 \tag{32}$$

where $K_i = J_i/kT$, $z_i = M_i/\langle a \rangle - 1$, $i = 1, 2$, and $M_1 = \langle bcd \rangle$, $M_2 = \langle abc \rangle$ for the Union Jack lattice. Now, after tracing over the four-coordinated sites, the Union Jack lattice becomes a free-fermion model (Wu and Lin 1987) with vertex weights

$$\begin{aligned}
\omega_1 &= 2\,\mathrm{e}^{K+K'} \cosh(K_1 + K_2 + K_3 + K_4) \\
\omega_2 &= 2\,\mathrm{e}^{-K-K'} \cosh(K_1 - K_2 + K_3 - K_4) \\
\omega_3 &= 2\,\mathrm{e}^{-K+K'} \cosh(K_1 - K_2 - K_3 + K_4) \\
\omega_4 &= 2\,\mathrm{e}^{K-K'} \cosh(K_1 + K_2 - K_3 - K_4) \\
\omega_5 &= 2 \cosh(K_1 - K_2 + K_3 + K_4) \\
\omega_6 &= 2 \cosh(K_1 + K_2 + K_3 - K_4) \\
\omega_7 &= 2 \cosh(K_1 + K_2 - K_3 + K_4) \\
\omega_8 &= 2 \cosh(-K_1 + K_2 + K_3 + K_4).
\end{aligned} \tag{33}$$

Using this equivalence Wu and Lin (1987) obtained the sublattice magnetisation

$$\langle a \rangle = I \tag{34}$$

where I is given by (21) with the ω given by (3) and (33). Now the tracing of the four-coordinated sites does not affect the three-spin correlations. Hence M_1, M_2, and z_1, z_2, of the Union Jack lattice are precisely those of the free-fermion model computed in the preceding section, provided that we use the ω weights (33). Solving (26) for $z_1 = M_1/I - 1$ and $z_2 = M_2/I - 1$ and substituting this solution into (32), we obtain after some reduction the final expression for the sublattice spontaneous magnetisation

$$\begin{aligned}
\langle \tau \rangle &= I[A_{1234}(K)(F_+ + F_-) + A_{2341}(K)(F_+ - F_-)] \\
A_{1234}(K) &= \sinh 2(K_1 + K_3)/\sqrt{2G_-(K) \sinh 2K_1 \sinh 2K_3}
\end{aligned} \tag{35}$$

with $F_\pm$ given by (27), $G_-(K)$ given by (23), and ω weights given by (3) and (33). This is the desired expression.

6. Summary and discussions

We have evaluated the three-spin correlation function for three Ising spins surrounding a unit cell in the free-fermion model. The results are given by (4) and (29). We have also obtained the spontaneous magnetisation of the Ising model on the Union Jack lattice with the most general anisotropic interactions, and the result is given by (35).

After the completion of this work we received a preprint from Professor Baxter and Dr Choy (Baxter and Choy 1989) in which they computed several local three-spin correlation functions, including M_1 and M_2, using a different route of transformations. We have verified that their results for M_1 and M_2 are identical with ours. Indeed, their S_1 is identically our M_1, and the identity can be seen by using the relation

$$F_+ + F_- = \frac{2\omega F}{\omega_1\omega_2}\left(\frac{\omega^2 - \omega_3\omega_4}{Q}\right)^{1/2} \qquad Q = G + 2\omega'^2 DV\sqrt{C} \tag{36}$$

where C is defined in (28) and D, G, F, V are those defined in (4.1) of Baxter and Choy (1988). It can also be shown that, using notation defined by (4.2) of Baxter and Choy (1988), the sublattice magnetisation $\langle\sigma\rangle$ is given by the compact expression, which agrees with (35) upon using the ω weights (33),

$$\langle\sigma\rangle = \frac{4I}{\omega_1\omega_2}\left(\frac{\sinh 2(J_1+J_3)F}{\sqrt{Q}} + \frac{\sinh 2(J_2+J_4)F^*}{\sqrt{Q^*}}\right) \tag{37}$$

where F^* is F (and Q^* is Q) with ω and ω' interchanged.

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